Viscous Flow and Heat Transfer Over a Permeable Shrinking Sheet with Partial Slip

REMUS-DANIEL ENE[1]*, MIHAI ALEXANDRU SZABO[2], SUZANA DANOIU[3]

[1] University Politehnica Timisoara, Department of Mathematics, 2 Victoriei Sq., 300006, Timisoara, Romania

[2] University Politehnica Timisoara, 2 Victoriei Sq., 300006, Timisoara, Romania

[3] Universitatea de Medicina si Farmacie Craiova, Facultatea de Medicina, 2-4 Petru Rares Str., 200349, Craiova, Romania

The viscous flow over a shrinking permeable sheet with partial slip is investigated. The flow is governed by a third-order nonlinear differential equation and heat transfer by a second-order differential equation. The equations of motion are solved analitically by Optimal Homotopy Perturbation Method (OHPM). This procedure is highly efficient and it controls the convergence of the approximate solutions. A few examples are presented, showing the exceptionally good agreement between the analytical and numerical solutions. OHPM is very efficient in practice, ensuring a very rapid convergence after only one iteration.

Keywords: Optimal homotopy perturbation method, viscous flow, partial slip, shrinking sheet,heat transfer.

The flow to a shrinking boundary with partial slip has yet become relevant in many situations and is very significant because of its several applications in engineering or industrial processes for example: glass fiber, crystal growing, paper production, drawing of electric films. Also, there are some situations where there may be a partial slip between the fluid and the boundary, e.g., the fluid may be a rarefied gas as mentioned by Sharipov and Selernev [1]. The no slip condition is replaced by Navier's partial slip condition, where the amount of relative slip is proportional to the local shear stress. The effect of stagnation slip flow on the heat transfer from a moving plate was recently considered by Wang [2]. Fang et al. [3] have solved the problem of viscous fluid flow model, without considering the heat transfer aspects, and they presented an exact solution of the governing Navier-Stokes equations. A very specific unsteady shrinking film solution was discussed by Wang [4]. The properties of the flow due to a shrinking sheet with suction are studied by Miklavčič and Wang [5]. Suction occurs when the fluid condenses on the surface, such as in chemical vapour deposition is considered by Jensen et al. [6]. The boundary layer flow and heat transfer over a permeable shrinking sheet with a first order slip flow model and the heat transfer rate at the surface are investigated by Aman and Ishak [7].

The objective of the present paper is to propose an accurate approach to nonlinear differential equation of a viscous fluid over a shrinking permeable sheet with partial slip and the heat transfer at the surface, using an analytical technique, namely optimal homotopy perturbation method [8-11]. The validity of our procedure, which does not imply the presence of a small or large parameter in the equation or into the boundary / initial conditions is based on the construction and determination of the auxiliary functions, combined with a convenient way to optimally control the convergece of the solution. The efficiency of our procedure is proved while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. The validity of this method is determined by comparing the results obtained with the results given by the numerical integration.

Equation of motion

Consider a two-dimensional laminar boundary layer flow over a shrinking boundary. If (u, v) is the velocity components in the Cartesian directions (x, y) and T is the temperature in the boundary layer, then the boundary layer equations are

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} \tag{2}$$

$$uT_x + vT_y = \alpha T_{yy} \tag{3}$$

where ν is the kinematic viscosity and α is the thermal diffusivity.

The partial slip boundary conditions are:

$$\begin{gathered} u = -ax + k\nu u_y, \quad v = V_w, \quad T = T_w, \quad at\ y = 0 \\ u \to 0, \quad T \to T_\infty \quad as\ y \to \infty \end{gathered} \tag{4}$$

where a>0 is the shrinking constant, k is a constant of proportionality, V_w is the mass transfer velocity at the surface of the sheet with $V_w>0$ for injection, $V_w<0$ for suction and $V_w=0$ for impermeable sheet, T_w is constant surface temperature.

In order to simplify the governing equations we use the similarity transformations:

$$\eta = \left(\frac{a}{\nu}\right)^{1/2} y, \quad \psi = (a\nu)^{1/2} xF(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{5}$$

where η is the independent similarity variable, $F(\eta)$ is the dimensionless stream function, $\theta(\eta)$ is the dimensionless temperature and ψ is the stream function defined as $u = \psi_y$ and $v = -\psi_y$ such that eq. (1) is automatically satisfied. From eq. (5) we get

$$u = axF'(\eta), \quad v = -(a\nu)^{1/2} F(\eta) \tag{6}$$

where prime denotes differention with respect to η.

* *email: remus.ene@upt.ro*

If we note s = F(0), then we take

$$V_w = -(av)^{1/2} s \tag{7}$$

Eqs (2) and (3) are reduced to the ordinary differential equations:

$$F'''(\eta) + F(\eta)F''(\eta) - [F'(\eta)]^2 = 0 \tag{8}$$

$$\theta''(\eta) + Pr\, F(\eta)\theta'(\eta) = 0 \tag{9}$$

where $Pr = \nu / \alpha$ is the Prandtl number.

Now, the boundary conditions (4) can be written in the form

$$F(0) = s, \quad F'(0) + 1 = \lambda F''(0), \quad F'(\eta) \to 0 \quad as\ \eta \to \infty \tag{10}$$

$$\theta(0) = 1, \quad \theta(\eta) \to 0 \quad as\ \eta \to \infty \tag{11}$$

where $\lambda = k\sqrt{a\nu}$ is a nondimensional parameter indicating the relative importance of partial slip. If $\lambda = 0$ there is no slip and if $\lambda \to \infty$, the surface is stress free [7].

Basic ideas of optimal homotopy perturbation method

We consider the following nonlinear differential equation

$$L(u) + N(u) = 0 \tag{12}$$

subject to the initial / boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0 \tag{13}$$

where L is a linear operator, N is a nonlinear operator and B is a boundary operator.

Generally speaking, the nonlinear operator N(u) depends on η, u , u', u'' and u''' and therefore one can write:

$$N(u) = G(\eta, u, u', u'', u''') \tag{14}$$

Applying the Taylor series theorem for real values α, β, γ, δ, we obtain

$$\begin{aligned} &G(\eta, u+\alpha, u'+\beta, u''+\gamma, u'''+\delta) = \\ &= G(\eta, u, u', u'', u''') + \frac{\alpha}{1!} G_u(\eta, u, u', u'', u''') + \\ &+ \frac{\beta}{1!} G_{u'}(\eta, u, u', u'', u''') + \frac{\gamma}{1!} G_{u''}(\eta, u, u', u'', u''') + \ldots \end{aligned} \tag{15}$$

We consider a homotopy, introducing a number of unknown auxiliary functions $K_{ij}(\eta, C_k)$, i, j, k = 1,2... that depend on the variable η and some parameters C_1, C_2, … which satisfies the following equation [8]:

$$\begin{aligned} &H(u,p) = L(u) + p[K_{11}(\eta, C_k)G(\eta, u_0, u_0', u_0'', u_0''') + \\ &+ K_{12}(\eta, C_k)G_u(\eta, u_0, u_0', u_0'', u_0''') + K_{13}(\eta, C_k)G_{u'}(\eta, u_0, u_0', u_0'', u_0''') + \\ &+ K_{14}(\eta, C_k)G_{u''}(\eta, u_0, u_0', u_0'', u_0''')] \end{aligned} \tag{16}$$

where $p \in [0, 1]$ is an embedding parameter, u_o is an initial guess of the solution $u(\eta, C_k)$ which satisfies the boundary conditions:

$$L(u_0) = 0, \quad B\left(u_0, \frac{\partial u_0}{\partial \eta}\right) = 0 \tag{17}$$

So, it is quite right to assume that the solutions of eq. (16) can be expressed as

$$u = u_0 + p u_1 + p^2 u_2 + \ldots \tag{18}$$

The first-order approximate solutions of eq. (12) can be readily obtained

$$\bar{u}(\eta, C_k) = u_0(\eta) + u_1(\eta, C_k) \tag{19}$$

where $u_1(\eta, C_k)$ is obtained by equating the coefficient of p for eq. (16). More precisely we have the following equation for u_1:

$$\begin{aligned} &L(u_1) + K_{11}(\eta, C_k)G(\eta, u_0, u_0', u_0'', u_0''') + \\ &+ K_{12}(\eta, C_k)G_u(\eta, u_0, u_0', u_0'', u_0''') + K_{13}(\eta, C_k)G_{u'}(\eta, u_0, u_0', u_0'', u_0''') + \\ &+ K_{14}(\eta, C_k)G_{u''}(\eta, u_0, u_0', u_0'', u_0'''), \quad B\left(u_1, \frac{\partial u_1}{\partial \eta}\right) = 0 \end{aligned} \tag{20}$$

In our procedure we can write only two equations: (17) and (20). The auxiliary functions K_{ij}, i = 1,2,...,5 are not unique and they can be chosen so that the products $K_{ij}G_a$ and G_a to be of the same form (a=u, u', u'' or u'''). The convergence-control parameters C_k, k = 1,2,..., which appear in the expression of the functions $K_{ij}(\eta, Ck)$ can be optimally determined. This can be done via various methods such as: the least squares method, the weighted residual, the collocation method, the Galerkin method and so on. The solutions of eqs. (12) and (13) can immediately be determined once the convergence-control C_k are known.

Application of OHPM to viscous flow and heat transfer given by eqs. (8-11)

We apply our procedure to obtain approximate solutions of eqs. (12) and (13). We choose the linear operator L by the form

$$L(F(\eta)) = F'''(\eta) + KF''(\eta) \tag{21}$$

where K > 0 is an unknown parameter.

Equation (17) becomes

$$\begin{aligned} &F_0'''(\eta) + KF_0''(\eta) = 0, \\ &F_0(0) = s, \quad F_0'(0) + 1 = F_0''(0), \quad F_0'(\infty) = 0 \end{aligned} \tag{22}$$

which has the following solution

$$F_0(\eta) = s + \frac{e^{-K\eta} - 1}{K(1 + \lambda K)} \tag{23}$$

The nonlinear operator is given from eqs (12), (8) and (21):

$$N(F(\eta)) = F(\eta)F''(\eta) - KF''(\eta) - [F'(\eta)]^2 \quad (= G(F, F', F'')) \tag{24}$$

From eq. (24) we obtain

$$\begin{aligned} &G_F(F, F', F'') = F''(\eta); \quad G_{F'}(F, F', F'') = \\ &= -2F'(\eta); \quad G_{F''}(F, F', F'') = F(\eta) - K \end{aligned} \tag{25}$$

Substituting eq. (23) into eqs (24) and (25) we have

$$\begin{aligned} &G(F_0) = \frac{\lambda s K^2 - \lambda K^3 + sK - K^2 - 1}{1 + \lambda K} e^{-K\eta}; \quad G_F(F_0) = \frac{K}{1 + \lambda K} e^{-K\eta}; \\ &G_{F'}(F_0) = \frac{2}{1 + \lambda K} e^{-K\eta}; \quad G_{F''}(F_0) = s - K + \frac{e^{-K\eta} - 1}{K(1 + \lambda K)} \end{aligned} \tag{26}$$

Equation (20) can be written in the form

$$F_1'''+KF_1''+K_{11}(\eta,C_k)\frac{\lambda sK^2-\lambda K^3+sK-K^2-1}{1+\lambda K}e^{-K\eta}+K_{12}(\eta,C_k)\frac{K}{1+\lambda K}e^{-K\eta}+$$
$$+K_{13}(\eta,C_k)\frac{2}{1+\lambda K}e^{-K\eta}+K_{14}(\eta,C_k)\left[s-K-\frac{1}{K(1+\lambda K)}+\frac{e^{-K\eta}}{K(1+\lambda K)}\right]=0,$$
$$F_1(0)=0,\quad F_1'(0)=\lambda F_1''(0),\quad F_1'(\infty)=0 \tag{27}$$

Having in view eq. (27), the auxiliary functions $K_{1i}(\eta, C_k)$, i = 1,2,3,4, must follow the terms in this equation, such that we can choose the functions $K_{1i}(\eta, C_k)$ in the following form:

$$K_{11}=K_{12}=K_{13}=0,$$
$$K_{14}=-(1+\lambda K)\left[(C_1+C_2\eta)e^{-K\eta}+(C_3+C_4\eta)e^{-2K\eta}+(C_5+C_6\eta)e^{-3K\eta}\right] \tag{28}$$

where C_j, j = 1,2,...,6, are unknown parameters at this moment.

Substituting eq. (28) into eq (27), we obtain equation in $F_1(_h)$

$$F_1'''+KF_1''=(AC_1+AC_2\eta)e^{-K\eta}+\left[C_1+AC_3+(C_2+AC_4)\eta\right]e^{-2K\eta}+$$
$$+\left[C_3+AC_5+(C_4+AC_6)\eta\right]e^{-3K\eta}+(C_5+C_6\eta)e^{-4K\eta}, \tag{29}$$

where $A = \lambda sK^2 - \lambda K^3 + sK - K^2 - 1$

Now, solving eq. (29) and then substituting this result and eq. (23) into eq. (19), we obtain the approximate solution of the first-order in the form:

$$\bar{F}(\eta,C_1,C_2,C_3,C_4,C_5,C_6)=s+\alpha-\frac{1}{K(1+\lambda K)}+\left[\frac{AC_2}{2K^2}\eta^2+\left(\frac{2AC_2}{K^3}+\frac{AC_1}{K^2}\right)\eta+\beta+\right.$$
$$\left.+\frac{1}{K(1+\lambda K)}\right]e^{-K\eta}-\left(\frac{C_2+AC_4}{4K^3}\eta+\frac{C_2+AC_4}{2K^4}+\frac{C_1+AC_3}{4K^3}\right)e^{-2K\eta}-\left(\frac{C_4+AC_6}{18K^3}\eta+\right.$$
$$\left.+\frac{7C_4+7AC_6}{108K^4}+\frac{C_3+AC_5}{18K^3}\right)e^{-3K\eta}-\left(\frac{C_6}{48K^3}\eta+\frac{5C_6}{288K^4}+\frac{C_5}{48K^3}\right)e^{-4K\eta}, \tag{30}$$

where

$$\alpha=-\frac{1+3\lambda K+8\lambda KA+4A}{4K^3(1+\lambda K)}C_1-\frac{1+2\lambda K+12\lambda KA+8A}{4K^4(1+\lambda K)}C_2-\frac{4+16\lambda K+27\lambda KA+9A}{36K^3(1+\lambda K)}C_3-$$
$$-\frac{8+20\lambda K+54\lambda KA+27A}{108K^4(1+\lambda K)}C_4-\frac{9+45\lambda K+64\lambda KA+16A}{144K^3(1+\lambda K)}C_5-\frac{27+120\lambda K+160\lambda KA+64A}{864K^4(1+\lambda K)}C_6 \tag{31}$$

$$\beta=\frac{1+45\lambda K+4\lambda KA+2A}{2K^3(1+\lambda K)}C_1+\frac{3+4\lambda K+12\lambda KA+8A}{4K^4(1+\lambda K)}C_2+\frac{1+3\lambda K+6\lambda KA+3A}{6K^3(1+\lambda K)}C_3+$$
$$+\frac{8+20\lambda K+54\lambda KA+27A}{108K^4(1+\lambda K)}C_4-\frac{9+45\lambda K+64\lambda KA+16A}{144K^3(1+\lambda K)}C_5-$$
$$-\frac{27+81\lambda K+160\lambda KA+64A}{864K^4(1+\lambda K)}C_6 \tag{32}$$

For eqs. (9) and (11), the linear operator and nonlinear operator are respectively:

$$L(\theta(\eta))=\theta''+h\theta',\quad N(\theta(\eta))=\theta'(Pr\,F-h)\quad (=G(\eta)) \tag{33}$$

where h is an unknown parameter.

In this case, eq. (17) becomes

$$\theta_0''+h\theta_0'=0,\quad \theta_0(0)=1,\quad \theta_0(\infty)=0 \tag{34}$$

and has the solution

$$\theta_0(\eta)=e^{-h\eta}(\tag{35}$$

Substituting eq. (35) into nonlinear operator G(θ) and $G_\theta(\theta)$, we obtain

$$G(\theta_0)=B_1e^{-h\eta}+B_2e^{-(h+K)\eta},\quad G_\theta(\theta_0)=B_3+B_4e^{-K\eta} \tag{36}$$

where

$$B_1=h^2-Pr\,s\,h+\frac{Pr\,h}{K(1+\lambda K)},\quad B_2=-\frac{Pr\,h}{K(1+\lambda K)},$$
$$B_3=Pr\,s-h-\frac{Pr}{K(1+\lambda K)},\quad B_4=\frac{Pr}{K(1+\lambda K)}. \tag{37}$$

Choosing the auxiliary functions $K_{2i}(\eta, D_k)$ in the form

$$K_{21}(\eta,D_k)=-(D_1+D_2h),\quad K_{22}(\eta,D_k)=-(D_3+D_4h)e^{-h\eta} \tag{38}$$

where Dj j = 1,2,3,4 are unknown parameters, equation from which we can obtain the first-approximation θ_1 is given by

$$\theta_1''+h\theta_1'=(B_1D_1+B_1D_2\eta)e^{-K\eta}+\left[B_2D_1+B_4D_3+(B_2D_2+\right.$$
$$\left.+B_4D_4)\eta\right]e^{-(K+h)\eta}+(B_3D_3+B_3D_4h)e^{-h\eta},\quad \theta_1(0)=\theta_1(\infty)=0 \tag{39}$$

The first-order approximate solution of eqs (9) and (11) can be obtain from the solution of eq. (39) and from eq. (35) and has the form

$$\bar{\theta}(\eta,D_i)=\theta_0+\theta_1=\left[\gamma+1-\left(\frac{B_3D_3}{h}+\frac{B_3D_4}{h^2}\right)\eta-\frac{B_3D_4}{2h}\eta^2\right]e^{-h\eta}+$$
$$+\left[\frac{(2K-h)B_1D_2}{K^2(K-h)^2}+\frac{B_1D_1}{K(K-h)}+\frac{B_1D_2}{K(K-h)}\eta\right]e^{-K\eta}+$$
$$+\left[\frac{B_2D_1+B_4D_3}{K(K+h)}+\frac{(2K+h)(B_2D_2+B_4D_4)}{K^2(K+h)^2}+\frac{B_2D_2+B_4D_4}{K(K+h)}\eta\right]e^{-(K+h)\eta}, \tag{40}$$

where

$$\gamma=-\frac{B_1D_1}{K(K-h)}+\frac{(h-2K)}{K^2(K-h)^2}B_1D_2-\frac{B_2D_1+B_4D_3}{K(K+h)}-\frac{(2K+h)(B_2D_2+B_4D_4)}{K^2(K+h)^2}$$

Numerical examples

We illustrate the accuracy of OHPM by comparing obtained approximate solutions with the numerical integration results obtained by means of a fourth-order Runge-Kutta method in combination with the shooting method. The unknown parameters are optimally identified via the least square method.

Example 5.1 In the first case, we consider s = 3, λ= 1, Pr = 0.71 and therefore eqs (32) and (40) respectively, can be whritten effectively in the form

$$\bar{F}(\eta)=2.9122291658+(0.0670048355-0.0291788256\,\eta+$$
$$+0.0037570554\eta^2)e^{-2.2751183999\eta}-(-0.0124569138-0.0330768587\eta)e^{-4.5502367998\eta}-$$
$$-(-0.0085355524-0.0060134134\eta)e^{-6.8253551997\eta}-$$
$$-(0.0002264677+0.0001971281\eta)e^{-9.1004735996\eta} \tag{41}$$

$$\bar{\theta}(\eta)=(-2.4170393387+0.4046215594\eta-0.0107990954\eta^2)e^{-2.1296349373\eta}+$$
$$+(3.4119615848+0.1488146412\eta)e^{-2.2751183999\eta}-$$
$$-(0.0050777538-0.0095415500\eta)e^{-4.4047533372\eta} \tag{42}$$

Example 5.2 In the last case, for s = 2, λ= 3, Pr = 11

$$\bar{F}(\eta)=0.3073746342+(3.4774214710-0.1378791520\eta+$$
$$+0.0177220631\eta^2)e^{-0.3839040237\eta}-(-4.4441933856+1.4747157246\eta)e^{-0.7678080474\eta}-$$
$$-(2.5607893726+2.2856461327\eta)e^{-1.1517120711\eta}-$$
$$-(3.6682001183+0.4569766849\eta)e^{-1.5356160948\eta} \tag{43}$$

$$\bar{\theta}(\eta)=(0.0830361216-0.0041430597\eta)e^{-0.3839040237\eta}+$$
$$+(-3.3280711666+0.8764027609\eta+0.1850908125\eta^2)e^{-1.9244601113\eta}+$$
$$+(4.2450350449+0.8544423327\eta)e^{-2.3083641350\eta} \tag{44}$$

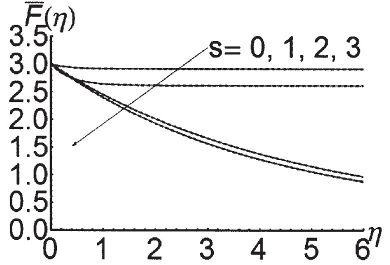

Fig. 1 Variation of $\bar{F}$ by increasing the partial slip parameter λ and $s=3$:
________ numerical solution;
----------- approximate solution

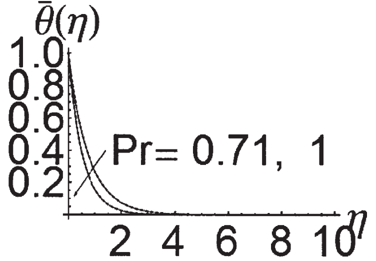

Fig. 2 Temperature profiles $\bar{\theta}$ for two values of Pr, and $s=2$, $\lambda=1$:
________ numerical solution;
----------- approximate solution;

In figures 1 and 2 are plotted a comparison between the first-order approximate solutions and numerical results obtained by means of the fourth-order Runge-Kutta method in combination with the shooting method.

Conclusions

In this work, the Optimal Homotopy Perturbation Method (OHPM) is employed to propose an analytic approximate solution to the viscous flow and heat transfer over a permeable shrinking sheet with partial slip parameter. Our procedure is valid even if the nonlinear equation of the motion does not contain any small or large parameters. The proposed approach is mainly based on a new construction of the solutions and especially on the involvement of the convergence-control parameters via the auxiliary functions. These parameters lead to an excellent agreement of the solutions with numerical results. This technique is very effective, explicit and accurate for nonlinear approximations rapidly converging to the exact solutions after only one iteration. The boundary layer equations governing the flow reduced to ordinary differential equations using a similarity transformation. These equations are solved to obtain the displacement and temperature distributions for various values of the partial slip parameter, mass suction parameter and Prandtl number. The effect of partial slip parameter and mass suction parameter, strongly influence the flow displacement and the temperature distribution in the boundary layer.

References

1. F. SHARIPOV, V. SELEZNEV, Data on internal rarefied gas flows, Journal of Physics and Chemical References Data, 27, 1998, 657–706.
2. C. Y. WANG, Stagnation slip flow and heat transfer on a moving plate, Chemical Engineering Science, 61, 2006, 7668–7672.
3. T. FANG, S. YAO, J. ZANG, A. AZIZ, Viscous flowover a shrinking sheet with a second order slip flow model, Communication in Nonlinear Science and Numerical Simulation, 15, 2010, 1831–1842.
4. C. Y. WANG, Liquid film on an unsteady stretching sheet, Quart. Appl. Math., 48, 1998, 601–610.
5. M. MIKLAVČIČ, C. Y. WANG, Viscous flow due to a shrinking sheet, Quart. Appl. Math., 64, 2006, 283–290.
6. K. F. JENSEN, E. O. EINSET, D. I. FOTIADIS, Flow phenomena in chemical vapor deposition in thin films, Ann. Rev. Fluid Mech., 23, 1991, 197–232.
7. F. AMAN, A. ISHAK, Boundary layer and heat transfer over a permeable shrinking sheet with partial slip, Journal of Applied Sciences Research, 6(8), 2010, 1111-1115.
8. V. MARINCA, N. HERIŞANU, Nonlinear Dynamical Systems in Engineering - Some Approximate Approaches, Springer Verlag, Heidelberg, Dordrecht, London, New York, 2011.
9. V. MARINCA, N. HERIŞANU, Optimal homotopy perturbation method for nonlinear differential equations, Nonlinear Sci. Lett. A, 1(3), 2010, 273–280.
10. V. MARINCA, N. HERIŞANU, Nonlinear dynamic analysis of an electrical machine rotor-bearing system by optimal homotopy perturbation method, Comput. Math. Appl., 61, 2011, 2019–2024.
11. REMUS-DANIEL ENE, Contributions on the extension of the Optimal Homotopy Asymptotic Method in solution of the flow of the polymeric materials, Mat. Plast., **51**, no. 2, 2014, p.194

Manuscript received: 27.11.2014